Theorising in mathematics education research: differences in modes and quality

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In mathematics education research reports, we find a bewildering array of "theories", "theoretical models" or "theoretical frameworks". The key notions and principles as well as the intellectual roots of these constructions are made more or less explicit, and the relations of theoretical entities to the empirical field under study are established in different ways. These differences imply discrepancies in quality. In this contribution we touch upon some of these issues. We attempt to show that an investigation of the relations between key concepts might help to read and evaluate theoretical underpinnings of research studies, and we argue that not all constructions that are labelled "theoretical" meet the criteria we consider essential for productive theorising. We also allude to different modes of engaging with empirical material and different ways in which theories are used in research studies. The main part of our discussion is limited to examples of "home-grown" theorising. The examples we have chosen to illustrate our points necessarily represent a biased selection.

As mathematics education research has evolved into an internationally acknowledged research field, methodological standards have been raised. The mere use of commonsense descriptions, the statement of didactical principles without reference to research, interpretations exclusively derived from introspection, or presentation of some quotes and examples of data that "speak for themselves" are no longer valued. However, the standards for carrying out and reporting research are by no means uniform. Even though we still might encounter journal articles that do not make explicit a theoretical underpinning of the research (Lerman, 2006),

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reference to some theory can be assumed to be a shared standard. However, what counts as "theory" is an issue of discussion within the field, as well as how theory is (or ought to be) used in research (Cobb, 2007; Mason & Waywood, 1996; Niss, 2007; Radford, 2008; Silver & Herbst, 2007). With the growth of the field, not only the range of phenomena of interest has widened, but also the variation in theories used has increased (Lerman, 2006; Prediger, Bikner-Ahsbahs & Arzarello, 2008). New grand theories have been imported into the field and many new local theories have emerged. Consequently researchers are producing more and more, often incommensurable, outcomes at very different levels of grain size of analysis. Educational studies in mathematics are at risk to become detached and isolated from each other. For beginning researchers this is particularly problematic, as choosing an appropriate branch of theorising is based mostly on tacit skills that are acquired only in the course of practising research. In some places PhD students are advised to use a distinct theory, but we also have anecdotical evidence for the suggestion to "throw away obsolete bodies of theorising and have a fresh start".

In our discussion, we want to draw attention to the differences in quality of what is disseminated in the field of mathematics education under the label of "theory", "theoretical model" or "theoretical framework" and to the variety of different modes by which these theories, theoretical models/ frameworks are established and used. As this can only be achieved by means of examples, restriction to a limited and biased selection is unavoidable. All examples concern modes of theorising from within mathematics education as a research domain, that is, attempts of "home-grown" theory building. While the examples are chosen because they are suitable to illustrate and discuss differences in mode and quality, the selection is arbitrary in relation to other aspects.

After shortly discussing the notion of theory, we point to differences in the relations between theoretical objects that are established in the examples we have chosen. As we will argue, not all constructions show "relational completeness" and not all are to the same extent related to previous research within the field; thus some do not resemble a theory.

For a discussion of the relation to the empirical field we draw on Bernstein's (2000) conception of external and internal languages of description. By means of examples, we show how these languages are more or less developed. We also touch upon different ways of employing or developing theory in research studies. In our discussion, in line with others, we suggest to pay serious attention to developments outside the field of mathematics education in order to advance theory.

What counts as theory?

Many attempts have been made to describe analytically the minimum ingredients of a conceptual system that account for its status as a theory. In accordance with a common view in empirical social sciences, which is also shared by researchers in mathematics education, a theory includes an organised system of theoretical entities, basic principles, and a relation to an empirical field in the form of a more or less explicitly developed methodology (see Niss, 2007; Radford, 2008). Theorising aims at making visible something that cannot be captured without mediation by the theoretical concepts. In this view, theorising aims at "producing understandings and ways of action" (Radford, 2008, p. 320). The characterisation also implies that a conceptual system gains the status of a theory only if it describes relationships between entities. Theories dealing with the same entities might establish different hierarchies amongst the basic concepts, depending on whether they, for example, privilege individual cognition over social interaction, reason over emotion, mind over body, or structure over agency.

As theories in mathematics education are not always presented in the form of an outline of an organised system of theoretical entities, basic principles, and an explicit methodology, these ingredients often can only be reconstructed by an analysis, which will often rely on knowledge about the intellectual roots and the historical development of a theory. In addition, agreement about what constitutes the empirical field of a given theory cannot always be assumed. To some extent, this will remain a matter of interpretation.

Types of relations between theoretical objects

The view that a theory establishes (hierarchical) relationships between entities (theoretical objects), as denoted above, suggests that a list of (possibly empirically derived) categories without a statement about how these are related does not count as theory. For the purpose of reconstructing the basic principles and identifying the explanatory potential of a theory, it is worthwhile to look at the types of relations between the basic concepts that are established.

We might find, for example, a *genus-species relation* (providing a hierarchical taxonomy if it includes several levels) or a *compositional relation* (where an entity is seen as composed of some parts). Such systems can be used for classification of empirical instances. The genus-species relation might be realised only in relation to the empirical, if the categories do not include sub-categories. Each empirical observation is then "an instance of" a general concept. The categories might be mutually exclusive or not. In mathematics education, such systems have been established for cataloguing students' errors in different mathematical areas. Movshovitz-Hadar, Zaslavksy and Inbar (1987), for example, identified six categories for classifying errors in high school mathematics: misused data, misinterpreted language, logically invalid inference, distorted theorem or definition, unverified solution and technical error. The initial classification was based on a content analysis of a large amount of empirical material, and then the categories were revised in order to establish mutual exclusiveness of the resulting categories. The relationship between the categories in this example is mutual exclusiveness and the categories are also exhaustive of the given data, but are not established prior to the analysis based upon some theory.

A compositional relation, in which one entity is a constituent part or a mode of another entity, is often visible in accounts of components of understanding particular mathematical concepts. For example, the components listed as constitutive for understanding fractions usually encompass part-whole comparison, measures, operators, quotients, and ratios and rates (e.g. Lamon, 2005). Constructions as those mentioned above are often labelled "a model of" or a "theoretical model of".

Another type of relationship between concepts is that of *antonymy*, as for example when "imitative reasoning" is described as the opposite of "creative reasoning" (Lithner, 2008), or, more prominently, in the distinction between relational and instrumental understanding (Skemp, 1976), or when holistic and atomic approaches to learning are conceptualised as antipodes (Svensson, 1984). A more complex relation between theoretical entities is, for example, the basis for the construction of "hypothetical learning trajectories" (Simon, 1995; see also e.g. Gravemeijer, Bowers & Stephan, 2003; Confrey et al., 2009) which are seen as "useful pedagogical, as well as theoretical, constructs" (Sarama & Clements, 2009, p. 17). The theoretical objects involved comprise "levels of understanding and skills" which are seen as occurring in a *timely order*, progressing towards more sophistication, in the course of the learner's development. An *action-product* relation is found in parts of the APOS theory (see below). *Causal relations* are rarely established by theories in mathematics education (and many would argue that this is neither possible nor desirable).

The following discussion of some examples investigates the relations between the set of concepts in constructions that are labelled theories, theoretical models, theoretical frameworks, etc. As we will argue, not all of these constructions in our view deserve the label theoretical.

Relations between key concepts – some examples

The four examples we will discuss have been selected, as pointed out above, only because they are suitable to illustrate and discuss differences in mode and quality of theorising. While the first example, the PISA framework, has been rather influential internationally, the second, more local example, has been chosen because the framework is claimed to constitute part of a new theory. We also consider modes of theorising which are utterly different to the first two examples, as these modes are of another calibre in terms of their intellectual roots and range, and their theoretical objects are of a much more complex nature. As one such example we analyse the APOS theory, and as another we discuss the *Antropological theory of didactics* (ATD), which accounts for a whole research programme.

The PISA framework

Even though the PISA is a survey and not a research study, we find it appropriate to shortly discuss the ingredients of what is called the "theoretical framework" of the PISA 2003 in a recent publication (OECD, 2009, p. 17) because this framework has initiated curriculum development in some countries and is used as theoretical background in research studies. The goal of the (permanent) survey is to empirically identify the degree of students' mathematical literacy. The mismatch between the theoretical roots and the framework for the PISA 2000, as reported in German publications, has been analysed by Gellert (2007). In the following we focus on the construction of the "theoretical framework". The assumptions and restrictions related to the measurement of mathematical literacy by means of a one-dimensional scale, resulting from the re-description of a general competence into a standardised performance measurement, has been discussed elsewhere (Jablonka, 2007). The empirical field, to which the PISA sample refers, is the mathematical literacy competency of all 15-year old students (attending educational institutions located within the country, in grades 7 and higher) in participating member countries and some partner countries and partner economies of the OECD. In the last round these were 25 451 204 individuals (calculated from table 11.1, pp. 178–180 in OECD, 2006). One *a priori* category by which this empirical field is structured is country-membership.

The theoretical framework first describes "mathematisation" as the main *constitutive* component of mathematical literacy (OECD, 2009, p. 20). In addition, eight (overlapping) mathematical competencies (cf. Niss, 1999) are listed as constitutive for mathematical literacy, as reflected in formulations such as "central to mathematical literacy", "critical" or "critically important" to mathematical literacy, or a "defining competency of mathematical literacy", an "important part of mathematical literacy" (OECD, 2009, p. 32). "Modelling" is one of these eight components. The description of modelling (p. 32) resembles that of mathematisation. By this, a constitutive category appears at the same time as a constitutive sub-category. This relation results in a circular construction.

In the course of the operationalisation in the form of test items, the descriptions of the eight competencies disappear, as the items are classified according to "the demands they placed on students' cognitive processing capabilities" (p. 37). In an exemplary description of some items from a previous round of the test, reference to some of the eight competencies is made selectively and only occasionally. The relation of the category "cognitive demand" to the eight constitutive competencies of mathematical literacy is explained as follows:

> These demands were identified by the competencies discussed and their amalgamation into the clusters of reproduction, connections, and reflection. (p. 37)

The necessity of the introduction of the three clusters is argued on the basis of practical demands:

> In order to productively describe and report student's capabilities, as well as their strengths and weaknesses from an international perspective, some structure is needed. (p. 33)

By this, the competency clusters are established as a form of an *a priori* operationalisation (without reference to empirical data) of the construct mathematical literacy. A hierarchical relation between the competencies in each cluster is implicitly suggested, that is, a student displaying a competency in the connections cluster needs to have developed the ones included in the reproduction cluster, and both of these are needed if the student can be said to display competencies in the reflection cluster.

In addition to competency clusters, the operationalised version of the theoretical framework includes a category named "overarching ideas", that is, a classification system for the assumed mathematical procedures, definitions, concepts etc. that are considered helpful in solving the tasks. However, these overarching mathematical ideas change into traditional names for the five sub-areas of school mathematical content used in the TIMSS (Grade 8) content classifications (Mullis, Martin, Gonzalez & Chrostowski, 2004) when used for the classification of test items.

Another dimension introduced for the classification of the PISA tasks in relation to mathematical literacy is the "context" to which a text of a test item refers: (i) personal, (ii) educational and occupational, (ii) public and (iii) scientific (including intra-mathematical). It is not clear on what conception of domains of social practice this list might be based. If ”scientific” refers to the domain of scientific activity that is available to a public audience (e.g. in newspaper reports), it is included in ”public”. ”Intra-mathematical” suggests that this is the domain of academic mathematics and not the public domain of mathematics. The category ”educational” must be interpreted as referring to mathematical activity in institutionalised education which was supposed not to be the focus of the test. The combination of ”educational” with diverse domains of occupational practices in which mathematical knowledge could be used, is unreasonable.

The three dimensions ”competencies”, mathematical ”content” and ”context” are not related. The theoretical framework appears as a conglomeration of didactical (mathematical literacy competencies), mathematical (content), cognitive (competency clusters) and common sense (context) concepts. In the course of the construction of the operationalised version, the hierarchy of the basic concepts changes.

Authentic tasks

The ”theory of authentic task situations” starts with the observation that school mathematics tasks have been criticised as being pseudo-realistic and that students have a tendency not to make proper use of their real-world knowledge when solving contextualised tasks. We have chosen this example because of its provenience in a Nordic country and also because it touches upon a problématique that is widely recognised in mathematics education. Palm (2009, p. 5) establishes the construction as ”a local theory of authentic task situations and a framework specifying one way of looking at the notion of authentic tasks” and as a contribution ”to a theoretical base” for the study of word problems that are ”realistic”, or ”authentic”. ”Task situations” are the practices indexed by a school mathematics word-problem given as a text (written or orally). The theory attempts to provide a ”fine-grained operational framework” for judging the ”concordance between in- and out-of-school situations” and includes some ”claims” (basic principles) about the fact that the enhancement of the authenticity (as described by the operational framework) of a word problem increases the ”the proportion of students that makes proper use of their real-world knowledge when working with a word problem” (p. 13).

The whole endeavour aims at comparing two different practices. Hence, the empirical field consists of (students solving) word-problems on the one hand, and ”out-of-school situations” in domestic or perhaps more specialised vocational practices on the other hand. As this is a comparatively large empirical field, we consider it as not only a ”local theory”, in contrast to its claim (see above). The operational framework constitutes a language for classifying word-problems given as school mathematics texts in terms of their ”representativeness” of out-of-school situations. The notion of representativeness is introduced by a reference to a work on performance measurement (Fitzpatrick & Morrison, 1971, pp. 237–240) and is seen as composed of ”comprehensiveness” and ”fidelity”. These three notions have to be seen as the fundamental concepts of the theory. Comprehensiveness denotes ”the range of different aspects of the situation that are simulated”, while fidelity refers to the ”degree to which each aspect approximates a fair representation in the criterion situation”. The operational framework consists of a list of aspects, which are seen as ”essential in the sense that their simulations clearly can affect the possibilities for the students to engage in the same mathematical activities in the school tasks as in the corresponding out-of-school situation” (Palm, 2002, p. 3). There are eight main aspects: event, question, information/data, presentation, solution strategies, circumstances, solution requirements and purpose in the figurative context; some of these are composed of sub-categories. Altogether, the framework comprises seventeen categories.

These categories are more or less visible in the school mathematics texts, but not necessarily in the corresponding situations, in which the (mathematical) knowledge employed is often tacit, the mathematical procedures as well as the data might remain unrecognized and the solution procedures rely on implicit conventions and can only be made explicit through research (see e.g. Gahamanyi, 2010). Hence, the categories provided cannot easily function as an operational framework for analysing the out-of-school situations that are indexed by a school mathematics word-problem in order to evaluate the representativeness of the latter.

As to the relationship between the categories of the framework, it remains unclear how these refer to the fundamental concepts of comprehensiveness and fidelity. Some of the aspects can only be judged as being present or absent in a word problem (such as a plausible ”event” and a plausible ”question”); thus these are not aspects of ”fidelity”. Others are a matter of degree (for example the sub-categories ”realism” and ”specificity” of ”information/data”) and can be seen as indicators of ”fidelity”. Hence, the framework consists of a list of categories with an (implicit) compositional, and in some aspects constitutive relation to the fundamental concept of representativeness. Whether there is a hierarchy between the aspects of a situation described in a word-problem in terms of their impact on students’ perceptions of the task is an empirical question. However, even though relations between the categories are not described in the framework, some of the aspects seem to be theoretically related: The presence/ absence or degree of one, influences the other. For example, the given information/data and the available tools ("circumstances") certainly affect the range of possible solution strategies, and if the "event" is fictive, then it does not make much sense to evaluate the representativeness of other aspects.

The descriptions of most of the categories are commonsensical and we spontaneously found some more categories that could be added to the list (such as the degree of freedom for changing the conditions for achieving the goal or the type of imagery in the school task). We also felt tempted to re-arrange the list of aspects on the base of a principle derived from some theory (for example in terms of social base, availability of symbolic and material tools, modes of division of labour etc., derived from some version of activity theory). We consider the mode of theorising in this example as relying on a strategy that is based on ad-hoc-constructions.

APOS

The following example relates to a branch of theories concerned with conceptual development in mathematics in terms of movements between engagement with mathematical operations or processes and mathematical objects, where the latter are constituted by the former through "reification" (Sfard, 1991) or "encapsulation" (Dubinsky, 1991; see also Pegg & Tall, 2010). As an example of this mode of theorising we consider the APOS theory, which sets out to be "a general theory of mathematical knowledge and its acquisition" (Dubinsky, 1991, p. 96). The focus is on mental constructions that can be made by the learners during instruction phases, and the theory is described as "an interpretation of Piaget's constructivism" (Cottrill et al., 1996, p. 171). The following quote from Piaget (provided by Dubinsky, 1991, p. 101) may serve to illustrate the generality of the fundamental principle in the theory:

> The whole of mathematics may therefore be thought of in terms of the construction of structures, [...] mathematical entities move from one level to another; an operation on such "entities" becomes in turn an object of the theory, and this process is repeated until we reach the structures that are alternatively structuring or being structured by "stronger" structures.

The theorising is based on three elements: (1) the "APOS cycle" (see figure 13 in Dubinsky, 1991, p. 107), mediated by the fundamental Piagetian concept of reflective abstraction; (2) the researcher's own mathematical understanding, including references to the historical evolution of mathematical entities; and (3) empirical observations of students working with mathematical entities. As a basic principle it is stated that mathematical knowledge occurs in three general types, actions, processes, and objects, organised in schemas (APOS). In the action phase, the response is controlling the individual, while during the transformation of mathematical objects (processes) the individual is in control of it. By reflecting on these processes they are encapsulated into new mathematical objects and thereby constructed. These objects can then also be de-constructed, that is, seen as processes. A schema is a coherent collection of actions, processes, and objects and other linked schemas. Reflecting on schemas is another way to construct new objects. The resulting objects can then be the starting point for a new cycle.

The relation established between the key concepts of the theory (actions, processes, and objects, organised in schemas) is one of reification, a process-into-product, a "thingifying" relation (with the connotation that the product is a real thing that exists). Reification is a comparatively general concept, used in a variety of intellectual fields. For example, the early Sanskrit grammarian Panini has generally claimed that all nouns are derived from verbs by such a process. As is well known, in Marxist theory reification (Verdinglichung) is a central concept, for example referring to the transformation of production and exchange value into use value. In mathematics (education), the concept of reification refers as its empirical field to both at the same time, the cognitive processes of individual learners of specific mathematical concepts as well as to the social development of mathematical sub-areas into systems of codified knowledge.

The APOS theory constitutes a comparatively high level of theorising with complex relationships between its key concepts. Some of the basic principles build on theorising outside the field of mathematics education.

The ATD

The Antropological Theory of Didactics (ATD) (Bosch & Gascon, 2006; Chevallard, 1997, 1998) proposes a theoretical framework that can be used for analysing mathematical practices in different institutions. As such, it might be classified as sociological theorising. A basic principle of the theory is the process of didactic transposition of mathematical knowledge (Chevallard, 1991; see also Bosch & Gascon, 2006), based on the assumption that knowledge selected to be taught in an educational institution has a pre-existence outside the institution, and in order to be teachable it has to be adapted depending on the constraints given in the didactic system (such as pre-knowledge of the students and the teachers, time, resources, and organisation). The didactic transposition has been described by three steps of transformation of knowledge, i.e. from the "scholarly knowledge" to "knowledge to be taught" to "taught knowledge" to "learned, available knowledge". A claim of the theory is that "the *minimum* unity [*sic*] of analysis of any didactic problem must contain all steps of the process of didactic transposition" (Bosch & Gascon, p. 56). Ligozat and Schubauer-Leoni (2009, pp. 2–3) characterize the didactic transposition process as

> (1) a decontexualisation of mathematical practices from the problems they originally attended, into [a] sequence of topics to fit the curricula constraints and the frames of teaching time; (2) a recontexualisation of these topics by the teachers, in order to make the students encounter the knowledge to be taught within the classroom practices.

In the ATD a number of theoretical notions are employed to capture different critical issues related to the didactic transposition process, such as transparency of knowledge, didactic time, chronogenèse, topogenèse, and disruption of mathematical objects (Chevallard, 1991). Some basic principles of the theory are established through statements about the extent to which these constrain and shape the outcome of the didactic transposition process.

Within the ATD, the theoretical object of a (mathematical) praxeology (or mathematical organisation) provides a unit of analysis (at different levels) for studies of the "ecology" of mathematical knowledge within institutions. The empirical field established by the ATD can thus be described as mathematical discourses in different institutions. A praxeology is seen to provide a general model of human activities in terms of a practical component (the know-how or praxis), and a discursive component (the know-why or logos) (Chevallard, 1997). In order to solve some type of tasks within an institution, appropriate techniques are developed. Although the distinction suggests a division of labour within an institution, the know-how is not seen as existing isolated from a discourse about why (justification or explanation) the chosen techniques apply, that is, a technology, which in turn is put into a wider context of meanings by reflections in terms of a theory. Another function of a technology is the production of (new) techniques. Theory plays the same role to technology as technology does to technique, i.e. as justification, explanation, or production. A specific constellation of a praxeology (in terms of tasks, techniques, technologies, theories) defines the structure of an institutionalised body of knowledge, such as a sub-field of mathematics (e.g. a calculus course at a university) or a part of a sub-field (e.g. theorems on continuous functions).

These theoretical objects (tasks, techniques, technologies, theories) are the constitutive components of a praxeology. In the ATD, hierarchical relations between these entities are suggested: Chevallard (1998) classifies praxeologies as point (*ponctuelle*), local, and regional. A given specific type of task defines a triplet of technique, technology, and theory: a point praxeology. A common technology for an aggregate of techniques for a set of types of tasks defines a local praxeology, while a set of technologies covered by one theory will specify a regional praxeology.

Empirical data are interpreted with respect to a "hierarchy of levels of co-determination" (Barbé, Bosch, Espinoza & Gascon, 2005, p. 256), a theoretical construct within the ATD which aims to provide a tool to analyse the relation between mathematical and didactical praxeologies at different levels of generality, from the most simple level "question" via "theme", "sector", "area", "discipline", "pedagogy", "school" to the most generic level, "society":

> The structure of a MO [mathematical organisation] at each level of the hierarchy determines the possible ways of organising its study and, reciprocally, the nature and the functions of a didactic organisation at each level determine, to a large extent, the kind of MOs that can be created (studied) in the considered institution.
>
> (Barbé et al., p. 256)

The theory thus provides principles to describe strong relationships between what is possible to teach and how to teach at different hierarchically ordered levels of determination.

The research programme linked to this theorising has developed a comparatively specialised language and cross-references between single studies are common. The ATD, though often only implicitly, also draws on a range of intellectual roots. Even if this name does not suggest so, the theory introduces a programme that sets out to develop a sociology of (mathematical) knowledge by studying "didactic systems". However, scholars working within the ATD have not, to our knowledge, engaged in a critical discussion of its basic principles and relations to other frameworks such as Activity Theory or to Bernstein's work that is concerned with the production, reproduction and distribution of knowledge, and in particular with the process of knowledge recontextualisation.

Different modes of "theorising"

The examples sketched above have been selected in order to illustrate different strategies of theorising in mathematics education. Inspired by Dowling's advanced method of describing two-dimensional spaces of strategies (see table 2 below), we could distinguish different modes of attempts of classifying, modelling or theorising. As with all research, mathematics education is discursive in nature and can only be understood in reference to previous research. However, the intertextuality can be more or less explicit (as for example by use of specialised language, references to intellectual roots, building on previous research outcomes). In the examples we discussed above, another dimension emerged, that is, the extent to which relations between the key concepts are established. We refer to this dimension as relational density. In table 1, the space created by these two dimensions is displayed.

Table 1. *Different modes of classifying, modelling or theorising*[1]

Intertextuality	Relational density	
	High	Low
High	**Theory**	**Conglomerate**
Low	**Local model**	**Ad-hoc construction**

In this categorisation, some of the examples discussed above resemble more of a conglomerate and of an ad-hoc construction, while others represent local models and theories. This is not to say that these other modes cannot be useful, but they cannot easily generate descriptions that generalise across contexts. We suggest using the term "theory" only for strategies that aim at relational completeness as well as interrogation and further development of previous research. In general, the internal explicitness and coherence and the relational completeness of the theory should serve as criteria for the evaluation of the analytic stage in a research study (see Dowling & Brown, 2010, p. 87).

A theoretical construct with low relational density between its basic concepts can have only a weak explanatory power. In an attempt to distinguish a collection of different re-interpretions of empirical phenomena from a theory proper, Moore (2006, in elaboration of Bernstein, e.g., 2000) argues that a theory describes a generating principle for a range of possibilities. It achieves this by generating *theoretical objects* that can be re-written in terms of empirical descriptions. A theoretical language makes it possible to outline configurations not yet observed in the empirical or experienced but untheorised and thus not recognised as empirical instances.

Moore uses the creation of the periodic table as an example. Mendeleev invented a principle that generated a two-dimensional scheme of positions for the elements (in terms of similar characteristics and of atomic weight). The point is that this matrix generated theoretical possibilities by predicting the existence and properties of new elements (only 63 of the 92 elements were recognised) and also suggested that the atomic weights attributed to some elements were wrong. As Moore points out, such theorising would provide the possibility of more than only re-describing the empirical by different languages that even might be based on incompatible approaches.

Amongst the examples discussed, the ATD might provide an example of this mode of theorising. The conceptualisation of mathematical discourses as a space of different constellations of the components of a praxeology allows the description of praxeologies that are not (yet) empirically realised.

However, not all conceptual systems that would fall under the category of theory in this schema, exhibit similar explicitness in terms of what is to count as an instance of a theoretical concept when encountered in the empirical. The price to be paid for a high level of theorising is the difficulty of the endeavour to describe an empirical field. In the following we further elaborate on this issue.

Relation to the empirical: languages of description

As to the relation to the empirical material, Radford (2008) stresses the coherence of the methodology with the basic principles and the ”operability” and also points to the unavoidable selectivity of data production and interpretation. An important point here is that a methodology incoherent with the basic principles will not produce relevant data and will not assist in data interpretation coherent with the theory. Methodology here refers to both, modes of data generation and of their interpretation. In the constraints discussed above for a conceptual system that might account for its status as theory, we did not include constraints on the relation to the empirical. It has been commonly argued that theories must be in principle testable or falsifiable, that they must help to make empirical data manageable, or provide descriptions of the empirical that enable understanding. Systems that do not bear any relation to the empirical sometimes also are labelled theoretical. However, we do not consider these here. The methodology mediates between research goals and questions posed in theoretical terms and the generating and reading of empirical material in relation to the questions, that is, between the questions and the outcomes of research.

Bernstein (2000) draws attention to the "discursive gap" between a theory (as a priori description) and the description related to the empirical material under study. He sees a close connection between theoretical model and methodology for data analysis (but not so much between the theoretical model and the ways in which data are collected, as for example through interviews or observations). For the purpose of advancing our discussion, we want to elaborate his distinction between what he calls "internal/external languages of description". A language of description is a "translation device whereby one language is transformed into another" (p. 132). What often is called theory or theoretical model, constitutes an internal language of description. It is a conceptual language with an explicit syntax that describes the relationships between conceptual entities. In order to describe "something other than itself", that is, empirical material, an external language of description has to be developed. It is a device for transforming observed empirical instances of a phenomenon of interest into theoretically relevant data. This point of view stresses the constructed (rather than collected) nature of "data", or the "text":

> A language of description constructs what is to count as an empirical referent, how such referents relate to each other to produce a specific text and translate these referential relations into theoretical objects or potential theoretical objects. In other words, the external language of description (L2) is the means by which the internal language (L1) is activated as a reading device or vice versa. A language of description, from this point of view, consists of rules for the unambiguous recognition of what is to count as a relevant empirical relation, and rules (realisation rules) for reading the manifest contingent enactments of these empirical relations. (Bernstein, 2000, p. 133)

This conception of the research process points to an open, dialectical relationship between the theoretical and the empirical and an emphasis on engaging in research that aims at theorising and not only at creating taxonomies of events and phenomena. From this point of view, in a research study, there ought to be a theoretical model, that is, an internal language (L1). For "reading" the empirical material in relation to the concepts and relations described by L1, there should be always choices, otherwise the adventure amounts to a rather closed view that does not allow theory development on the base of empirical data. The development of the L2 should be as independent as possible from the L1, that is, when confronted with empirical material, one should try to ignore the theoretical model and try to model the data with a view on the potential space, that is, the repertoire of sense making on the side of the participants. Then, the question of possible ”readings” of these descriptions help creating an external language of description. As there are always points of choice, the reliability can be enhanced by making these points explicit. Limits of the L1 might be revealed as well as restrictions stemming from the assumptions made. This then initiates theory development (refinement of the L1). Development of related internal and external languages of description might be achieved in a research programme rather than in a single study.

Dowling (1998) describes the external and internal language of description (syntax) in terms of the strength of their ”discursive saturation” (short DS) and provides a two-dimensional characterisation of different strategies of engaging with the empirical that could be linked to different scientific disciplines. But the characterisation is also useful on a micro-level for describing (in their extreme forms) different constructions that constitute the bewildering array of ”theories”, ”theoretical models”, ”theoretical frameworks” etc. in the field of mathematics education research.

Table 2. *Grammatical modes* (Dowling, 2007, p. 4; layout adjusted)

External syntax (gaze)	Internal syntax	
	DS+	DS-
DS+	**Metonymic apparatus**	**Method**
DS-	**Metaphoric apparatus**	**Fiction**

The ”gaze” encompasses the principles that determine the constitution of the empirical objects. ”Discursive saturation” is a dimension that describes the extent to which a practice, as for example the practice of developing scientific theories, has explicit principles of regulation: a high extent is noted DS+, a low DS-. The ”discursive” is the domain of the linguistic actions of a practice. Craft, for example, is a practice that usually does not include much explicit explanation and the utterances are highly context dependent. Such a practice is described as having low discursive saturation, that is, ”there is a low degree of saturation of the non-discursive by the discursive” (Dowling, 1998, p. 30). If the regulation of a practice lies more within the linguistic (implying that the meanings are less context-dependent), these practices are described to exhibit a high discursive saturation ”because there is a relatively high level of saturation of the non-discursive by the discursive” (p. 32). DS+ is not always to be confused with accessibility (e.g. a long chain of logical inferences). Also, in research there are practices that exhibit a relatively low discursive saturation, such as conducting open interviews, or carrying out qualitative data analysis, or finding a mathematical proof. The distinction between DS+ and DS- is one of relative saturation because no practice (not even mathematics) can be fully realised within discourse (cf. Dowling, 2009, p. 104).

In the two-dimensional categorisation above, a "fiction" cannot make precise empirical claims because the external language of description is under-developed and in addition there is a weak syntax of theoretical language. On the other hand, a "metonymic apparatus" has an explicit syntax for both the internal and the external language, as for example some parts of physics (with clearly defined observational languages for measuring empirical instances). Such apparatus, however, appears to be quite inflexible in terms of further development. If the external language is under-developed, the reference to something that counts as an empirical instance of a theoretical object is metaphorical. A well-developed language for describing empirical material, perhaps without the aspiration of developing (grand) theory, is called a "method".

Languages of description – some examples

To illustrate and discuss the relation between internal and external languages of description in the context of research in mathematics education, we will use two theories presented above as examples, the APOS theory and the ATD.

APOS

In relation to its empirical field, the APOS theory does not set out to scrutinise its core claim of the mental construction of knowledge as a reification process. As a local theory about the acquisition of specific mathematical concepts, it is informed by the empirical in the form of outcomes of instruction. It also includes a theory of instruction. As a key methodological device to plan instruction and develop the (local) theory, a "genetic decomposition" of the mathematical concept in focus is used. This is defined as "a tool we can use to make sense of data relating to a student's understanding of a concept" (Cottrill et al., 1996, p. 189). As an example, a genetic decomposition of the limits of functions is developed in Cottrill et al. (pp. 177–178) consisting of the following seven steps (other examples of genetic decompositions are found in Dubinsky, 1991):

1. The action of evaluating f at a single point x that is considered to be close to, or even equal to a.
2. The action of evaluating the function f at a few points, each successive point closer to a than was the previous point.

3. Construction of a coordinated schema as follows.
(a) Interiorization of the action of Step 2 to construct a domain process in which x approaches a.
(b) Construction of a range process in which y approaches L.
(c) Coordination of (a), (b) via f. That is, the function f is applied to the process of x approaching a to obtain the process of $f(x)$ approaching L.

4. Perform actions on the limit concept by talking about, for example, limits of combinations of functions. In this way, the schema of Step 3 is encapsulated to become an object.

5. Reconstruct the processes of Step 3(c) in terms of intervals and inequalities. This is done by introducing numerical estimates of the closeness of approach, in symbols, $0 < |x - a| < \delta$ and $|f(x) - L| < \varepsilon$.

6. Apply a quantification schema to connect the reconstructed process of the previous step to obtain the formal definition of a limit.

7. A completed $\varepsilon - \delta$ conception applied to specific situations.

This description provides an (a priori) operationalisation of the theory into a prescription (in terms of key concepts of the theory) for how to learn a specific mathematical concept. It can be seen as the part of the internal language of description that informs the external language of description on how to read the empirical observations, that is how to interpret them as research data. To be able to develop the external language of description, it is necessary to place the genetic decomposition within the research cycle advocated by the theory: a theoretical perspective is applied on the mathematical topic chosen, both related to its historical development and to individuals' learning processes; a genetic decomposition is constructed; based on the genetic decomposition, an instructional sequence is designed and implemented; students are observed and interviewed; the outcomes of the observations are analysed in terms of the theory; a (revised) genetic decomposition is constructed, etc.

The instructional design is open in relation to the theory. In the case of limits of functions in Cottrill et al. (1996), instruction included a focus on a variety of computer activities, followed by classroom tasks (without computers), discourse, and exercises. This design was a choice made by the researchers informed by arguments based on their interpretations of the theory. By this choice, the external language of description was strongly influenced by the internal when the empirical observations were transformed into empirical data by searching for indicators that students who succeeded in the tasks seemed to be interiorizing a particular action into a process or encapsulating a certain process to an object, and by confirming that these indicators were not present in those students who did not succeed. Such influence is often inevitable but one should, not only for the sake of the validity of the research outcomes but also for ethical reasons, struggle to minimize it:

> [T]he external description, irrespective of the translation demands of L1 (the model), must as far as possible, be permeable to the potential enactments of those being described. Otherwise their voice will be silenced. (Bernstein, 2000, p. 135)

By the cyclic research process in the example discussed here, the researchers' attempt to let data take precedence over theory may seem to counteract this risk:

> When a step in the genetic decomposition does not appear to arise in the data, and if we feel that the questions we asked did relate to the particular issue in the step, then we may drop it from the genetic decomposition. This is not a problem if the step arose mainly from theoretical considerations. If, however, it arose from data in previous experiments, then it becomes necessary to revise the genetic decomposition so as to be consistent with data from all experiments, past and present. (Cottrill et al., 1996, p. 169)

However, as it is only the specific genetic decomposition that is open to revision, the internal language of description will retain its precedence over the external language of description, and the empirical research is imposing the theory on the subjects studied. By the research process, the external language of description (i.e. the realisation rule by way of the genetic decomposition) remains hidden in the tasks and the interview questions, and it is only the researchers, and not the participating subjects (the students), who have access to the recognition rule [2]. This kind of phenomenon is, according to Bernstein (2000), inherent in research using experimental design.

The ATD

As one part of its empirical realisations, researchers employing the ATD as a theoretical framework set out to describe the specific mathematical praxeologies that can be observed in a certain institution at different levels of the didactic transposition process. A purpose for such analysis is to study how institutional and didactic restrictions affect teachers' practice. As an example, in Barbé et al. (2005) it was shown, in the context of limits of functions in the Spanish high school, how the mathematical praxeology observed at the level of knowledge to be taught, due to a complex historical process of didactic transposition, was "split" between a "topology of limits" and an "algebra of limits", where the former was seen to be historically rooted in scholarly questions about the nature and existence of limits, and the latter in the problem of how to calculate (algebraically) the limit for a given family of functions (in the high school mainly quotients of polynomials). As the practical part of the praxeology by tradition focused only on the algebra of limits with no link to the theoretical part outlining a topology of limits, the observed mathematical praxeology at the level of knowledge actually taught, the teacher in this study, as well as the students, had problems to explain, justify and give meaning to the work on limits, "distortions" on the classroom practice that are due to constraints coming from the first steps of the didactic transposition.

These conclusions were based on empirical material consisting of syllabi and textbooks, and classroom observations, questionnaires, teachers' and students' notes and interviews. The reading of the empirical data material was mediated by a language employing key notions from the theory and a description of the didactic process in the classroom in terms of six "moments" (Barbé et al., p. 238). This external language of description organised the empirical data in one table outlining a "transcript of the teaching process", and one table with an "analysis of the teaching process" (pp. 247–248), thus describing the didactical praxeology by which the mathematical praxeology was set up. The number and sequential order of the didactic moments are independent of the theory but described by way of its key notions and motivated by stating that "each moment has a specific function to fulfil which is essential for a successful completion of the didactic process" (p. 238). To be able to describe the mathematical praxeologies, a "reference mathematical organisation" was constructed by the researchers, by way of an analysis of current syllabi documents and textbooks, which constituted their

> epistemological model of the "scholarly knowledge" that legitimates the knowledge to be taught. It is the broader map with reference to which we can interpret the mathematical contents that are proposed to be studied at school. (p. 241)

These different elements of the external language of description make it possible to read (and analyse) the empirical data in terms of the theory, such as describing the types of tasks, techniques and technologies used in the institution (for another example, see Hardy, 2009). However, the language does not provide an unambiguous description of the objects to analyse as it employs the (ambiguous) categories of the study process, in terms of the postulated six moments, and the derived reference mathematical organisation. In addition, the data are interpreted with respect to the hierarchy of levels of co-determination, the categories of which are not clearly delineated to the extent that ambiguous interpretations of the data are avoided. This top-down approach with respect to the role of theory sets constraints to establishing an open, dialectical relationship between the theoretical and the empirical, and thereby delimits empirically grounded theory development.

How theory is used

In research practice, theory use may vary both with respect to the extent to which the research refers to a particular theory and to the attitude towards the theory. One may *use the whole theory*, that is, asking a paradigmatic research question, adopting the set of basic principles and methodology, and/or delimiting and modelling the objects studied using the tools of the theory (as for example the APOS in Cottrill et al., 1996, and the ATD in Barbé et al., 2005). There are also cases where the researchers do not employ the whole "package" but confine themselves to what could be called a *pitching* on particular key concepts for the purpose of orienting oneself towards the object of study, thus using the concepts of the theory as thinking tools (as for example in da Ponte & Marques, 2007, who employ some of the item classification categories used in the PISA framework; or in Selter, 2009, where reference is made to the notion of the didactical contract to account for students' word problem solving behaviour). Yet other modes include the use of theory that may consist of mainly *employing analytical tools* that often have their origin in a particular branch of theorising in mathematics education or in a related social science (such as critical discourse analysis, see e.g. Le Roux, 2008), or as *general background* in asking research questions that bear the spirit of a theory (such as the mere stating that a study will adopt a socio-cultural perspective).

Concerning attitudes to theories, there are at least two main ways of "reading" a theory. In a *dogmatic reading* there is an explicit reference to the intellectual roots of the approach and to the set of basic propositions. This results in the obligation to also take up the key concepts (and perhaps also the methodology). With a *creative or "heretic" misreading* one is trying to specify and develop, modify, or reject a theory and/or (some of) its concepts. A dogmatic reading of the ATD is found in, for example, Barbé et al. (2005). The study by Hardy (2009) provides an example of a creative misreading of the same theory, as it is not simply run over the data but pointing to possible theory development by combining the ATD with a complementary framework of institutions. Similarly, the *Joint action theory in didactics* is an example of an attempt to overcome some weaknesses of the ATD and the TDS (*The theory of didactic situations*, as developed by Guy Brousseau) by employing an integrative approach (Ligozat & Schubauer-Leoni, 2009).

Bernstein (2000) points to the issue of a reduced research economy, in the British context, and as a consequence lack of time, as an explanation for a decrease of proper theorising in educational research. Also in the Nordic context the increased number, in recent years, of doctoral theses presented as collections of papers contribute to this trend. A support for employing such less developed theoretical frameworks is found in recent methodological literature advocating the use of so called "conceptual frameworks" (or *bricolage*) as appropriate in mathematics education research (e.g. Cobb, 2007; Lester, 2005).

Discussion

The constraints theories put on the methodologies differ with the extent to which they contain an external language of description. In summary, the examples of proper theories (in contrast to ad-hoc constructions and conglomerates) discussed above contain more or less specified external languages of description and set up different relations to the empirical. They include key concepts, basic principles and define what constitutes a problématique. The relation to the empirical can be prescriptive or descriptive. In the example of APOS, a constructivist theory of learning in undergraduate mathematics, there are at least two basic principles: (1) All mathematical entities can be analysed in terms of actions, processes, objects, and schemas, and (2) Individuals deal with mathematical problem situations by constructing mental actions, processes, objects and organizing them in schemas (in this order). This theory is a prescriptive model for constituting teaching units and a descriptive model for failure/ success of students on a task in relation to their specific mental constructions. The method for constructing the empirical (teaching units) is unspecified, as well as the method for gathering data and analysing the empirical (students' solutions). This theory does not include a specified external language of description, which implies low validity for the interpretation of the data. We find a normalising effect of prescriptive use of theories of instruction: the model of the empirical reality eventually becomes what it intended to analyse (if everyone would teach according to the theory).

By the analyses and the examples presented above, we want to emphasise that, in our view, theorising, with high relational density and intertextuality (see table 1), is both an essential component and a goal of research. As pointed out by Dowling and Brown "[s]ome form of theory is absolutely essential" (2010, p. 87). Atheoretical research (that is, research based on *conglomerates* and *ad-hoc constructions*, see table 1) might amount only to collections of unconnected re-descriptions of empirical phenomena. Such cataloguing could be termed "botany". However, it is only by means of *theorising* that singular events with apparent differences can be related to general principles. General principles allow seeing these different events as exemplars, and at the same time define the relevant qualities they have in common. Theories that include generating principles can depict possible empirical realisations that have not yet been observed.

In our discussion, we only provided examples of home-grown theorising, that is, of theories that emerged from within the field of mathematics education. This does not necessarily mean that these do not draw on theories from outside the field; the reference might just remain implicit (in the case of the APOS theory such reference is explicit). Further, home-grown theories have the potential of productively interacting with theories developed outside the field of mathematics education, if resemblances are noted. The notion of didactic transposition in the ATD, for example, echoes Bernstein's notion of recontextualisation (see e.g. Bernstein, 2000). If references are explicit, the combination and mutual amendment of theories sharing common intellectual roots is facilitated. This could be fruitful especially in cases when a theory with an underdeveloped external language is supplemented by a methodology from a cognate branch of theorising. By this, development of contradictory conglomerates can be obviated.

However, many researchers in mathematics education explicitly employ and adapt theories from other fields without attempting to develop genuine theories of phenomena related to the teaching and learning of mathematics. As researchers in mathematics education usually are not working in sociology, or social anthropology, psychology, philosophy, history of mathematics, or linguistics, there might be a problem of being not fully acquainted with the traditions of theorising in those fields. Consequently it might be hard to decide which sets of key propositions, concepts, methodologies to draw on and to combine, without producing a contradictory framework for analysis. Theories and methodological devices taken from other fields inevitably have to be adapted to the peculiarities of mathematics education. Some of their complex theoretical insights and relations might get distorted or lost in this process. However, by only taking into account home-grown theories as a resource in a research study, there is a danger of doing so without taking notice of the profound theorising that has been developed over long periods in philosophy, psychology, linguistics, anthropology and sociology. By saying this, we want to stress (in line with Steiner, 1985; see also Sriraman & English, 2010) the importance of paying serious attention to developments outside the field of mathematics education in order to advance theory. Also, thinking about relationships between different theories, can be assisted by drawing on thinking about those relationships in other areas.

For the purpose of synthesising outcomes of research in mathematics education, it is necessary to consider the consistency and affinity of the basic propositions of the underpinning theories. If researchers in mathematics education draw on well-developed theories from other fields, this is not problematic, if the theories are not fragmented into too many pieces before imported, and if the meaning of basic notions borrowed is not distorted beyond recognition. Without paying attention to the theoretical consistency of research, educational studies in mathematics are at risk to become more detached and isolated from each other.

References

Barbé, J., Bosch, M., Espinoza, L. & Gascon, J. (2005). Didactical restrictions on the teacher's practice: the case of limits of functions in Spanish high schools. *Educational Studies in Mathematics*, 59, 235–268.

Bergsten, C. & Jablonka, E. (2009, January). *Interpreting students' reasoning through the lens of two different languages of description: integration or juxtaposition*? Paper presented at CERME6, Lyon.

Bernstein, B. (2000). *Pedagogy, symbolic control and identity. Theory, research and critique* (Revised edition). Oxford: Rowman & Littlefield.

Bosch, M. & Gascon, J. (2006). 25 years of didactic transposition. *ICMI Bulletin*, 58, 51–65.

Chevallard, Y. (1991). *La transposition didactique du savoir savant au savoir enseigné*. Grenoble: Editions Pensée Sauvage.

Chevallard, Y. (1997). *Familière et problématique, la figure du professeur.* Texte issu d'un cours donné à la VIIIe école d'été de didactique des mathématiques, Saint-Sauves, 22–31 août 1995. (Also published 1997 in *Recherches en Didactique des Mathématiques*, 17 (3), 17–54.)

Chevallard, Y. (1998). *Analyse des pratiques enseignantes et didactique des mathématiques: l'aproche anthropologique*. In Actes de l'université d'été d'IREM de Clermont-Ferrand 'Analyse des pratiques enseignantes et didactique des mathématiques', La Rochelle, 4–11 juillet 1998, p. 91–120. (Also published 1999 in *Recherches en Didactique des Mathématiques*, 19 (2), 221–266)

Cobb, P. (2007). Putting philosophy to work. Coping with multiple research perspectives. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 3–37). Charlotte, NC: Information Age Publishing.

Confrey, J. Maloney, A., Nguyen, K., Mojica, G. & Myers, M. (2009). Equipartitioning/splitting as a foundation of rational number reasoning using learning trajectories. In M. Tzekaki, M. Kaldrimidou & C. Sakonidis (Eds.), *Proceedings of the 33rd conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 1–8). Thessaloniki: PME.

Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K., et al. (1996). Understanding the limit concept: beginning with a coordinated process scheme. *Journal of Mathematical Behavior*, 15, 167–192.

Dowling, P. (1998). *The sociology of mathematics education: mathematical myths/pedagogic texts*. London: Falmer Press.

Dowling, P. (2007). Organising the social. *Philosophy of Mathematics Education Journal*, 21. Retrieved March 30, 2010 from http://people.exeter.ac.uk/PErnest/pome21

Dowling, P. (2009). *Sociology as method: departures from the forensics of culture, text and knowledge*. Rotterdam: Sense Publishers.

Dowling, P. & Brown, A. (2010). *Doing research / reading research. Re-interrogating education* (2nd ed.). London: Routledge.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 95–123). Dordrecht: Kluwer.

Fitzpatrick, R. & Morrison, E. J. (1971). Performance and product evaluation. In R. L. Thorndike (Ed.), *Educational measurement* (2nd ed.) (pp. 237–270). Washington, DC: American Council on Education.

Gahamanyi, M. (2010). *Mathematics at work. A study of mathematical organisations in Rwandan workplaces and educational settings* (Doctoral thesis. Linköping Studies in Behavioural Science No. 150). Department of Behavioural Sciences and Learning, Linköping University.

Gellert, U. (2007). Mathematik "in der Welt" und mathematische "Grundbildung" – zur Konsistenz der mathematikdidaktischen Rahmens von PISA. In T. Jahnke & W. Meyerhöfer (Eds.), *Pisa & Co. Kritk eines Programms* (2. erweiterte Auflage) (pp. 375–390). Berlin: Franzbecker.

Gravemeijer, K., Bowers, J. & Stephan, M. (2003). A hypothetical learning trajectory on measurement and flexible arithmetic. In M. Stephan, J. Bowers, P. Cobb & K. Gravemeijer (Eds.), *Supporting students' development of measuring conceptions: analyzing students' learning in social context* (Journal for Research in Mathematics Education. Monograph, Vol. 12) (pp. 51–66). Reston, VA: NCTM.

Hardy, N. (2009). Students' perceptions of institutional practices: the case of limits of functions in college level calculus courses. *Educational Studies in Mathematics*, 72 (3), 341–358.

Jablonka, E. (2007). Mathematical literacy: die Verflüchtigung eines ambitionierten Testkonstrukts. In T. Jahnke & W. Meyerhöfer (Eds.), *Pisa & Co. Kritk eines Programms* (2. erweiterte Auflage) (pp. 247–280). Berlin: Franzbecker.

Lamon, S. (2005). *Teaching fraction and ratios for understanding*. Mahwah, NJ: Lawrence Erlbaum Associates.

Lerman, S. (2006). Theories of mathematics education: is plurality a problem? *Zentralblatt für Didaktik der Mathematik*, 38 (1), 8–13.

Le Roux, K. (2008). A critical discourse analysis of a real-world problem in mathematics: looking for signs of change. *Language and Education*, 22 (5), 307–326.

Lester, F. (2005). On the theoretical, conceptual, and philosophical foundations for research in mathematics education. *Zentralblatt für Didaktik der Mathematik*, 37, 457–467.

Lithner, J. (2008). A research framework for creative and imitative reasoning. *Educational Studies in Mathematics*, 67 (3), 255–276.

Ligozat, F. & Schubauer-Leoni, M.-L. (2009, January). *The joint action theory in didactics: Why do we need it in the case of teaching and learning mathematics?* Paper presented at CERME6, Lyon.

Mason, J. & Waywood, A. (1996). The role of theory in mathematics education research. In A. Bishop, K. Clements, C. Keitel, J. Kilpatrick & C. Laborde (Eds.), *International handbook of mathematics education* (pp. 1054–1089). Dordrecht: Kluwer Academic Publishers.

Moore, R. (2006). Knowledge structures and intellectual fields: Basil Bernstein and the sociology of knowledge. In R. Moore, M. Arnot, J. Beck & H. Daniels (Eds.), *Knowledge, power and educational reform: applying the sociology of Basil Bernstein* (pp. 28–43). New York: Routledge.

Movshovitz-Hadar, N., Zaslavksy, O. & Inbar, S. (1987). An empirical classification model for errors in high school mathematics. *Journal for Research in Mathematics Education*, 18 (1), 3–14.

Mullis, I.V.S., Martin, M. O., Gonzalez, E. J. & Chrostowski, S. J. (2004). *TIMSS 2003 international mathematics report: findings from IEA's Trends in International Mathematics Science Study at the fourth and eighth grades*. Chestnut Hill, MA: Boston University International TIMSS Study Center.

Niss, M. (1999). Kompetencer og uddannelsesbeskrivelser. *Uddannelse*, 9, 21–29.

Niss, M. (2007). Reflections on the state of and trends in research on mathematics teaching and learning: from here to Utopia. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 1293–1312). Charlotte, NC: Information Age Publishing.

OECD. (2006). *PISA 2006 technical report*. Paris: Author.

OECD. (2009). *Learning mathematics for life: a perspective from PISA*. Paris: Author.
Palm, T. (2002). *The realism of mathematical school tasks: features and consequences* (Doctoral thesis). Department of Mathematics, University of Umeå.
Palm, T. (2009). Theory of authentic task situations. In L. Verschaffel, B. Greer, W. van Dooren & S. Mukhopadhyay (Eds.), *Words and worlds. Modelling verbal descriptions of situations* (pp. 3–19). Rotterdam: Sense Publishers.
Pegg, J. & Tall, D. (2010). The fundamental cycle of concept construction underlying various theoretical frameworks. In B. Sriraman & L. English (Eds.), *Theories of mathematics education. Seeking new frontiers* (pp. 173–192). Berlin: Springer.
Ponte, J. da & Marques, S. (2007). Proportion in school mathematics textbooks: a comparative study. In D. Pitta-Pantazi & G. Philippou (Eds.), *European research in mathematics educaiton V, proceedings of CERME5* (pp. 2443–2452). University of Cyprus.
Prediger, S., Bikner-Ahsbahs, A. & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: first steps towards a conceptual framework. *ZDM – The International Journal on Mathematics Education*, 40 (2), 165–178.
Radford, L. (2008). Connecting theories in mathematics education: challenges and possibilities. *ZDM – The International Journal on Mathematics Education*, 40 (2), 317–327.
Sarama, J. & Clements, D. (2009). *Early childhood mathematics education research. Learning trajectories for young children*. New York: Routledge.
Selter, C. (2009). Stimulating reflection on word problems by means of students' own productions. In L. Verschaffel, B. Greer, W. van Dooren & S. Mukhopadhyay (Eds.), *Words and worlds. Modelling verbal descriptions of situations* (pp. 315–332). Rotterdam: Sense Publishers.
Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics Education*, 22, 1–36.
Silver, E. & Herbst, P. (2007). Theory in mathematics education scholarship. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 39–67). Charlotte, NC: Information Age Publishing.
Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, 26, 114–145.
Skemp, R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching*, 77, 20–26.
Sriraman, B. & English, L. (2010). Surveying theories and philosophies of mathematics education. In B. Sriraman & L. English (Eds.), *Theories of mathematics education. Seeking new frontiers* (pp. 7–32). Berlin: Springer.

Steiner, H.-G. (1985). Theory of mathematics education (TME): an introduction. *For the Learning of Mathematics*, 5 (2), 11–17.
Svensson, L. (1984). Skill in learning and organising knowledge. In F. Marton, D. Hounsell & N. Entwistle (Eds.), *The experience of learning: implications for teaching and studying in higher education* (pp. 59–71). Edinburgh: Scottish Academic Press.

Notes

1 The category ”relational density” used in table 1 describes an aspect of discursive saturation (see below on this term).

2 This phenomenon is evidenced in Bergsten and Jablonka (2009), by comparing the outcomes of interpretations of the same data from the point of view of two different theoretical orientations.

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